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On some geometric constant and the extreme points of the unit ball

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Abstract

Mitani and Saito introduced a geometric constant $\gamma_{X,\psi}$ by using the notion of ψ-direct sum. For $t\in[0,1]$, the constant $\gamma_{X,\psi}$ is defined as a supremum taken over all elements of in the unit sphere. It is proved that for a Banach space X with a predual Banach space X_{*}, $\gamma_{X,\psi}$ can be calculated as the supremum can be taken over all extreme points of the unit ball.

1. Introduction

Let X be a Banach space with $\dim X\geq 2$. By S_{X} and B_{X}, we denote the unit sphere and the unit ball of X, respectively. The von Neumann-Jordan constant (shortly, NJ constant) $C_{NJ}(X)$ is defined as the smallest constant C for which

$$\frac{1}{C}\leq\frac{\|x+y\|^{2}+\|x-y\|^{2}}{2(\|x\|^{2}+\|y\|^{2})}\leq C$$

holds for all $x,y\in X$, not both zero (Clarkson [2]). This constant has been considered in many papers. It is known that $1\leq C_{NJ}(X)\leq 2$ for any Banach space X. From the parallelogram law it follows immediately that X is a Hilbert space if and only if $C_{NJ}(X)=1$ ([3]). Recall that a Banach space X is uniformly non-square provided $C_{NJ}(X)<2$ ([10]), where X is said to be unifomly non-square if there exists $\delta>0$ such that $\|x+y\|\leq 2(1-\delta)$ holds whenever $\|x-y\|\geq 2(1-\delta)$, $x,y\in S_{X}$.

By the definition, the NJ constant is in the following form;

$$C_{NJ}(X)=\sup\left\{\frac{\|x+y\|^{2}+\|x-y\|^{2}}{2(\|x\|^{2}+\|y\|^{2})}:x,y\in X,(x,y)\neq(0,0)\right\},$$

and it can be reformulated as

$$C_{NJ}(X)=\sup\left\{\frac{\|x+ty\|^{2}+\|x-ty\|^{2}}{2(1+t^{2})}:x,y\in S_{X},0\leq t\leq 1\right\}.$$

In 2006, the function γ_{X} was introduced by Yang and Wang [13];

$$\gamma_{X}(t)=\sup\left\{\frac{\|x+ty\|^{2}+\|x-ty\|^{2}}{2}:x,y\in S_{X}\right\},\quad(t\in[0,1]).$$

It is easy to see that the NJ constant $C_{NJ}(X)$ coincide with $\sup\{\gamma_X(t)/(1+t^2) : 0 \le t \le 1\}$. Thus, the function γ_X is useful to calculate the NJ constant $C_{NJ}(X)$ for some Banach spaces. In fact, they computed $C_{NJ}(X)$ for X being Day-James spaces ℓ_∞-ℓ_1 and ℓ_2-ℓ_1 by using the function γ_X.

In the same paper [13], they noted that, for a finite dimensional Banach space the supremum can be taken over all extreme points of the unit ball. We obtained a generalization of this.

2. Preliminaries

Recall that a norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be *absolute* if $\|(x,y)\| = \|(|x|,|y|)\|$ for all $(x,y) \in \mathbb{R}^2$, and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. The ℓ_p-norms $\|\cdot\|_p$ $(1 \le p \le \infty)$ are basic examples;

$$\|(x,y)\|_p = \begin{cases} (|x|^p + |y|^p)^{1/p} & \text{if } \ 1 \le p < \infty, \\ \max\{|x|,|y|\} & \text{if } \ p = \infty. \end{cases}$$

The family of all absolute normalized norms on $\mathbb{R}^2$ is denoted by AN_2. As in Bonsall and Duncan [1], AN_2 is in a one-to-one correspondence with the family Ψ_2 of all convex functions ψ on $[0,1]$ with $\max\{1-t,t\} \le \psi(t) \le 1$ for all $0 \le t \le 1$. Indeed, for any $\|\cdot\| \in AN_2$ we put $\psi(t) = \|(1-t,t)\|$. Then $\psi \in \Psi_2$. Conversely, for all $\psi \in \Psi_2$ let

$$\|(x,y)\|_\psi = \begin{cases} (|x|+|y|)\psi\left(\dfrac{|y|}{|x|+|y|}\right) & \text{if } \ (x,y) \neq (0,0), \\ 0 & \text{if } \ (x,y) = (0,0). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_2$, and $\psi(t) = \|(1-t,t)\|_\psi$ (cf. [8]). The functions corresponding to the ℓ_p-norms $\|\cdot\|_p$ on $\mathbb{R}^2$ are given by

$$\psi_p(t) = \begin{cases} \{(1-t)^p + t^p\}^{1/p} & \text{if } \ 1 \le p < \infty, \\ \max\{1-t,t\} & \text{if } \ p = \infty. \end{cases}$$

In [11], the notion of ψ-direct sum of Banach spaces was introduced. Let X and Y be Banach spaces, and let $\psi \in \Psi_2$. The ψ-direct sum $X \oplus_\psi Y$ of X and Y is defined as the direct sum $X \oplus_\psi Y$ equipped with the norm

$$\|(x,y)\|_\psi = \|(\|x\|,\|y\|)\|_\psi \quad ((x,y) \in X \oplus Y).$$

This notion has been investigated by several authors.

In [5], Mitani and Saito introduced a geometrical constant $\gamma_{X,\psi}$ of a Banach space X, by using the notion of ψ-direct sum. For a Banach space X and $\psi \in \Psi_2$, the function $\gamma_{X,\psi}$ on $[0,1]$ is defined by

$$\gamma_{X,\psi}(t) = \sup\{\|(x+ty, x-ty)\|_\psi : x,y \in S_X\}, \quad (t \in [0,1]).$$

One can easily have $\gamma_X(t) = \gamma_{X,\psi_2}(t)^2/2$, where $\psi_2 \in \Psi_2$ is the function which corresponds to the ℓ_2-norm $\|\cdot\|_2$.

Mitani and Saito showed that

Proposition 2.1. ([5])

(1) *For any Banach space* X, $\psi \in \Psi_2$ *and* $t \in [0,1]$,

$$2\psi\left(\frac{1-t}{2}\right) \leq \gamma_{X,\psi}(t) \leq 2(1+t)\psi\left(\frac{1}{2}\right).$$

(2) *For a Banach space* X, $\psi \in \Psi_2$ *and* $t \in [0,1]$,

$$\gamma_{X,\psi}(t) = \sup\left\{\|(x+ty, x-ty)\|_\psi : x, y \in B_X\right\}.$$

(3) *Let* $\psi \in \Psi_2$ *with* $\psi \neq \psi_\infty$. *Then a Banach space* X *is uniformly non-square if and only if* $\gamma_{X,\psi}(t) < 2(1+t)\psi(1/2)$ *for any (or some)* t *with* $0 < t \leq 1$.

They obtained some other results on $\gamma_{X,\psi}$ (cf. [5]).

3. Results

An element $x \in S_X$ is called an extreme point of B_X if $y, z \in S_X$ and $x = (y+z)/2$ implies $x = y = z$. The set of all extreme points of B_X is denoted by $\mathrm{ext}(B_X)$.

In [13], Yang and Wang noted that

Proposition 3.1. *Let* X *be a finite dimensional Banach space. Then*

$$\gamma_X(t) = \sup\left\{\frac{\|x+ty\|^2 + \|x-ty\|^2}{2} : x, y \in \mathrm{ext}(B_X)\right\}.$$

There exists some infinite-dimensional Banach spaces whose unit ball has no extreme point. However, from the Banach-Alaoglu Theorem and Krein-Milman Theorem, we have that for any Banach space, the unit ball of the dual space is the weakly* closed convex hull of its set of extreme points.

For $\psi \in \Psi_2$, the dual function ψ^* of ψ is defined by

$$\psi^*(s) = \sup_{t\in[0,1]} \frac{(1-s)(1-t)+st}{\psi(t)}$$

for $s \in [0,1]$ ([4]). Then we have $\psi^* \in \Psi_2$ and that $\|\cdot\|_{\psi^*}$ is the dual norm of $\|\cdot\|_\psi$. It is easy to see that $\psi^{**} = \psi$.

Suppose that X is a Banach space with the predual Banach space X_*. Then the unit ball B_X is the weakly* closed convex hull of $\mathrm{ext}(B_X)$, and the direct sum $X \oplus_\psi X$ is isomorphic to the dual of $X_* \oplus_{\psi^*} X_*$. Based on these facts, we obtain the following result.

Theorem 3.2. [6] *Let* X *be a Banach space with the predual Banach space* X_*. *Then*

$$\gamma_{X,\psi}(t) = \sup\left\{\|(x+ty, x-ty)\|_\psi : x, y \in \mathrm{ext}(B_X)\right\}$$

for any $\psi \in \Psi_2$ *and any* $t \in [0,1]$.

In [9], Takahashi introduced the James and von Neumann-Jordan type constants of Banach spaces. For $p \in [-\infty, \infty)$ and $t \geq 0$, the James type constant is defined as

$$J_{X,p}(t) = \begin{cases} \sup\left\{ \left(\dfrac{\|x+ty\|^p + \|x-ty\|^p}{2} \right)^{1/p} : x, y \in S_X \right\} & \text{if } p \neq -\infty, \\ \sup\{\min(\|x+ty\|, \|x-ty\|) : x, y \in S_X\} & \text{if } p = -\infty \end{cases}$$

(cf. [12, 14]). The von Neumann-Jordan type constant is defined as

$$C_p(X) = \sup\left\{ \frac{J_{X,p}(t)^2}{1+t^2} : 0 \leq t \leq 1 \right\}.$$

For $p \in [1, \infty)$ and $t \in [0, 1]$, it is easy to see that $J_{X,p}(t) = 2^{-1/p}\gamma_{X,\psi_p}(t)$. Thus we have the following results on the James and von Neumann-Jordan type constants.

Corollary 3.3. *Let X be a Banach space with the predual Banach space.*

(1) *For any $p \in [1, \infty)$ and any $t \in [0, 1]$,*

$$J_{X,p}(t) = \sup\left\{ \left(\frac{\|x+ty\|^p + \|x-ty\|^p}{2} \right)^{1/p} : x, y \in \mathrm{ext}(B_X) \right\}$$

(2) *For any $p \in [1, \infty)$,*

$$C_p(X) = \sup\left\{ \frac{(\|x+ty\|^p + \|x-ty\|^p)^{2/p}}{2^{2/p}(1+t^2)} : x, y \in \mathrm{ext}(B_X), 0 \leq t \leq 1 \right\}.$$

In particular, on the modulus of convexity and the NJ constant, one can easily have

$$\rho_X(t) = J_{X,1}(t) - 1 = \frac{\gamma_{X,\psi_1}(t)}{2} - 1$$

for any $t \in [0, 1]$, and

$$C_{NJ}(X) = C_2(X) = \sup\left\{ \frac{\gamma_{X,\psi_2}(t)^2}{2(1+t^2)} : 0 \leq t \leq 1 \right\}.$$

Hence we obtain

Corollary 3.4. *Let X be a Banach space with the predual Banach space. Then,*

$$\rho_X(t) = \sup\left\{ \frac{\|x+ty\| + \|x-ty\|}{2} - 1 : x, y \in \mathrm{ext}(B_X) \right\}$$

for all $t \in [0, 1]$, and

$$C_{NJ}(X) = \sup\left\{ \frac{\|x+ty\|^2 + \|x-ty\|^2}{2(1+t^2)} : x, y \in \mathrm{ext}(B_X),\ 0 \leq t \leq 1 \right\}.$$

4. Examples

For p,q with $1 \le p,q \le \infty$, the Day-James space ℓ_p-ℓ_q is defined as the space $\mathbb{R}^2$ with the norm

$$\|(x_1,x_2)\|_{p,q} = \begin{cases} \|(x_1,x_2)\|_p & \text{if} \quad x_1x_2 \ge 0, \\ \|(x_1,x_2)\|_q & \text{if} \quad x_1x_2 \le 0. \end{cases}$$

Yang and Wang [13] calculated the von NJ constant of the Day-James spaces ℓ_∞-ℓ_1 and ℓ_2-ℓ_1 by using the notion of $\gamma_X(t)$.

Remark that ℓ_∞-ℓ_1 and ℓ_2-ℓ_1 have the predual spaces ℓ_1-ℓ_∞ and ℓ_2-ℓ_∞, respectively (cf. [7]). Thus, from Theorem 3.2, we obtain

$$\gamma_{X,\psi}(t) = \sup\{\|(x+ty, x-ty)\|_\psi : x,y \in \mathrm{ext}(B_X)\}$$

for X being ℓ_∞-ℓ_1 or ℓ_2-ℓ_1.

Example 4.1. *Let X be the Day-James space ℓ_∞-ℓ_1, $\psi \in \Psi_2$ and $t \in [0,1]$. Then*

$$\mathrm{ext}(B_X) = \{\pm(1,1),\ (\pm1,0),\ (0,\pm1)\},$$

and hence

$$\gamma_{X,\psi}(t) = (2+t)\max\left\{\psi\left(\frac{1}{2+t}\right), \psi\left(\frac{1+t}{2+t}\right)\right\}.$$

Example 4.2. *Let X be the Day-James space ℓ_2-ℓ_1, $\psi \in \Psi_2$ and $t \in [0,1]$. Then*

$$\mathrm{ext}(B_X) = \{(x_1,x_2) : x_1^2 + x_2^2 = 1, x_1x_2 \ge 0\},$$

and so

$$\gamma_{X,\psi}(t) = (1+t+\sqrt{1+t^2})\max\left\{\psi\left(\frac{1+t}{1+t+\sqrt{1+t^2}}\right), \psi\left(\frac{\sqrt{1+t^2}}{1+t+\sqrt{1+t^2}}\right)\right\}.$$

We note that some geometric constants does not necessarily coincide with the supremum taken over all extreme points of the unit ball. The constant

$$C_Z(X) = \sup\left\{\frac{\|x+y\|\|x-y\|}{\|x\|^2+\|y\|^2} : x,y \in X,\ (x,y) \ne (0,0)\right\}.$$

was introduced by Zbăganu [15]. As in the von Neumann-Jordan constant, this constant is reformulated as

$$C_Z(X) = \sup\left\{\frac{\|x+ty\|\|x-ty\|}{1+t^2} : x,y \in S_X,\ 0 \le t \le 1\right\}.$$

Example 4.3. *Let X be the Day-James space ℓ_∞-ℓ_1. Then*

$$\sup\left\{\frac{\|x+ty\|\|x-ty\|}{1+t^2} : x,y \in \mathrm{ext}(B_X),\ 0 \le t \le 1\right\} < C_Z(X).$$

From [9], the Zbăganu constant $C_Z(X)$ coincide with the von Neumann-Jordan type constant $C_0(X)$. Hence, for any $\psi \in \Psi_2$, the Zbăganu constant $C_Z(X)$ can not be expressed by the means of $\gamma_{X,\psi}$.

For any q less than 1, can we obtain a Banach space X in which the von Neumann-Jordan type constant $C_q(X)$ does not coincide with the supremum taken over all extreme points of the unit ball B_X ?

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